

Thermal phase slips in superconducting films

Mikhail A. Skvortsov

*Landau Institute for Theoretical Physics,
Chernogolovka, Russia*

In collaboration with

Artem V. Polkin

supported by
the RSF under Grant
No. 23-12-00297

Landau Week
Yerevan, June 24, 2025

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Superconducting Nanowire Single-Photon Detectors

**Herschel Space Observatory
(2009—2013)**



Operates at 1.4 K

dirty superconductor inside

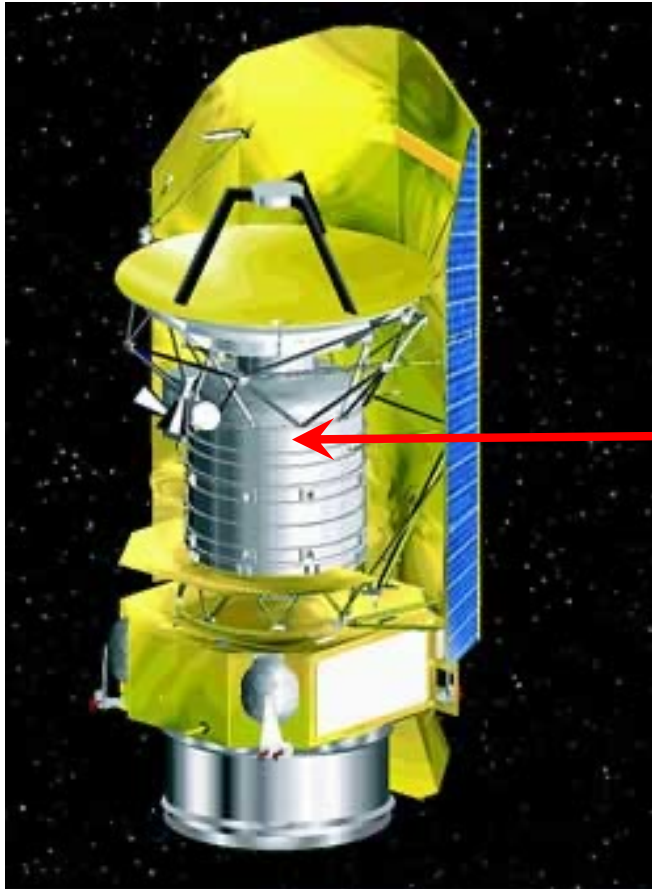
Thought to be
star formation in the early Universe



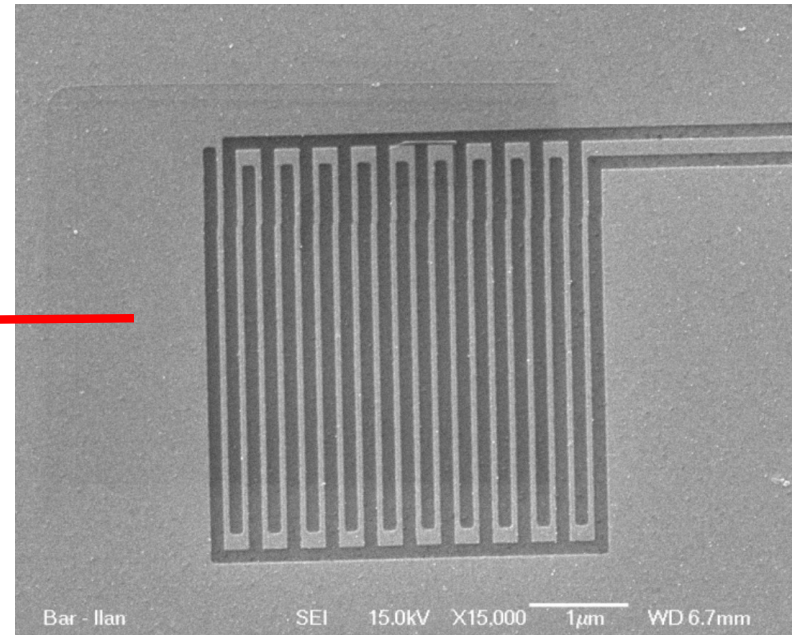
far infrared and submillimeter
wavebands (55–672 μm)

Superconducting Nanowire Single-Photon Detectors

**Herschel Space Observatory
(2009—2013)**



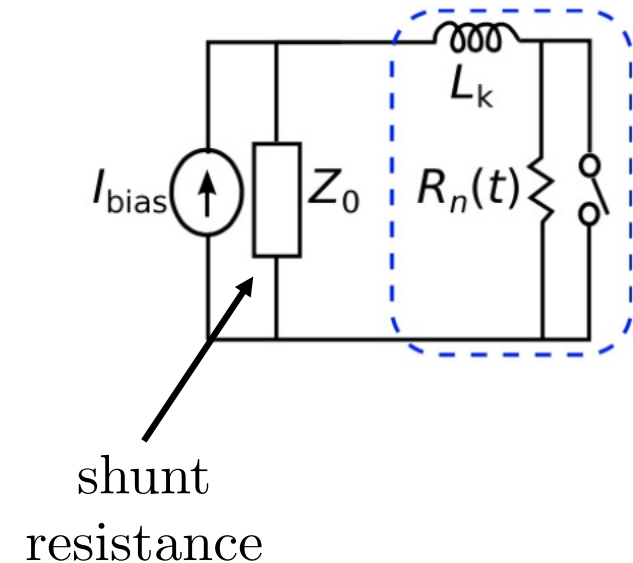
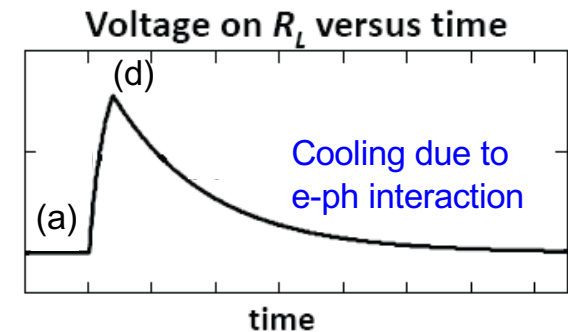
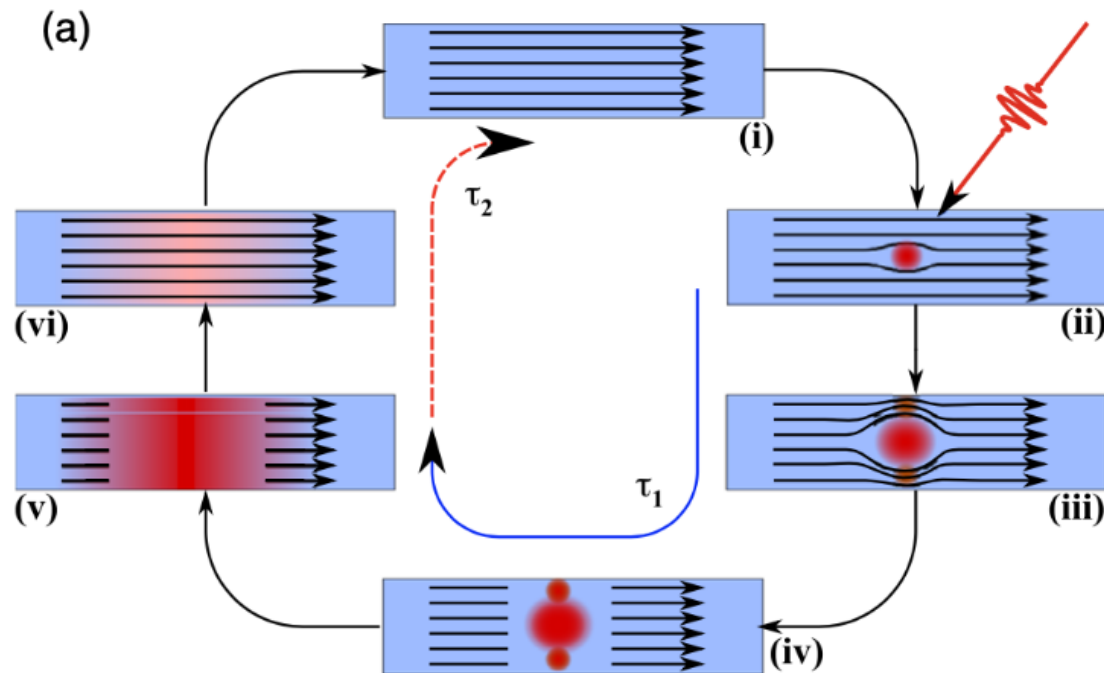
Operates at 1.4 K



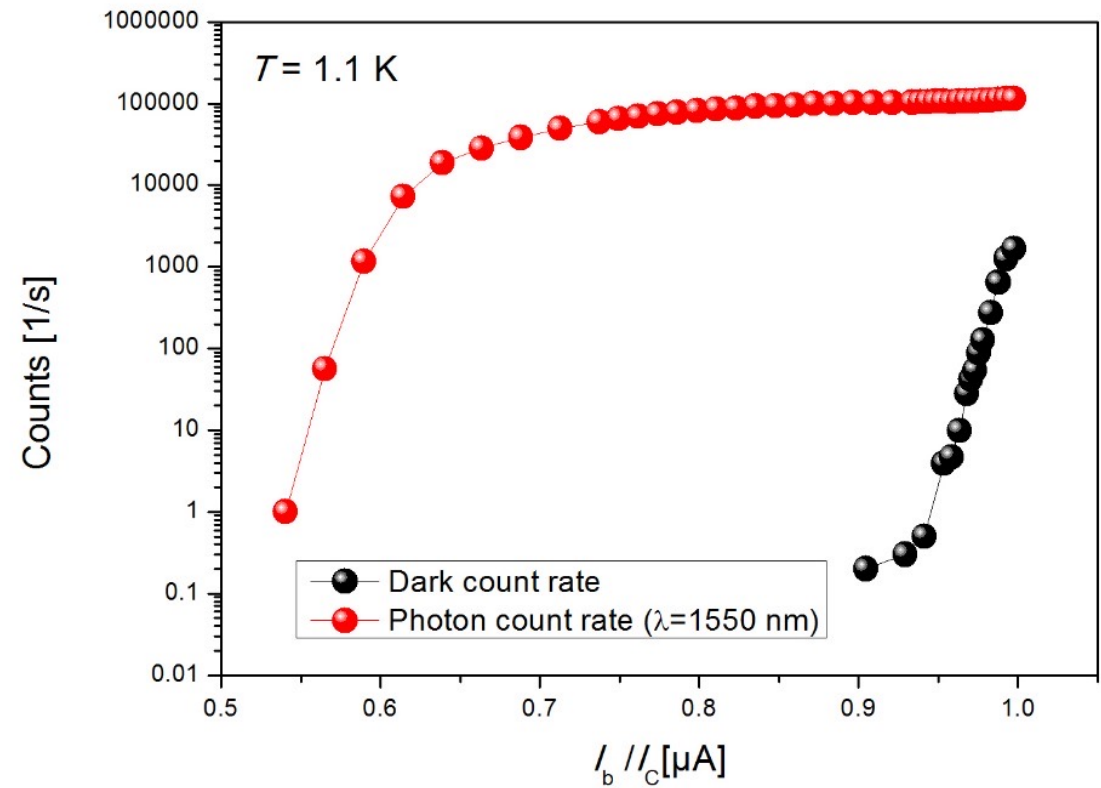
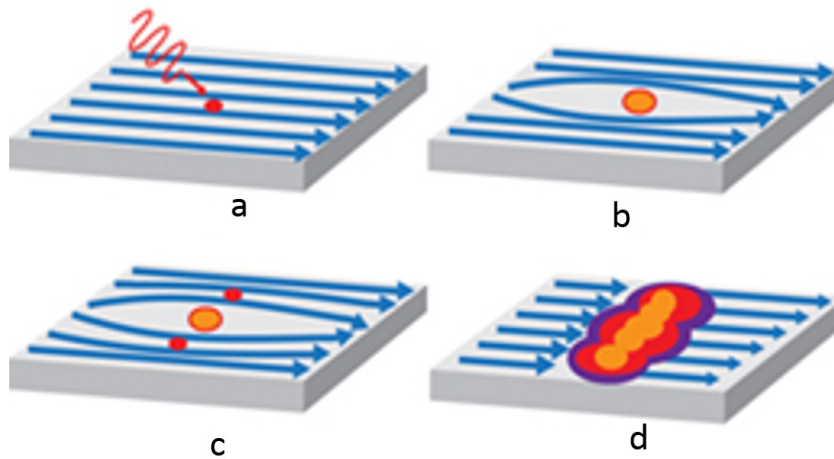
meandered NbN wire

Superconducting Nanowire Single-Photon Detectors

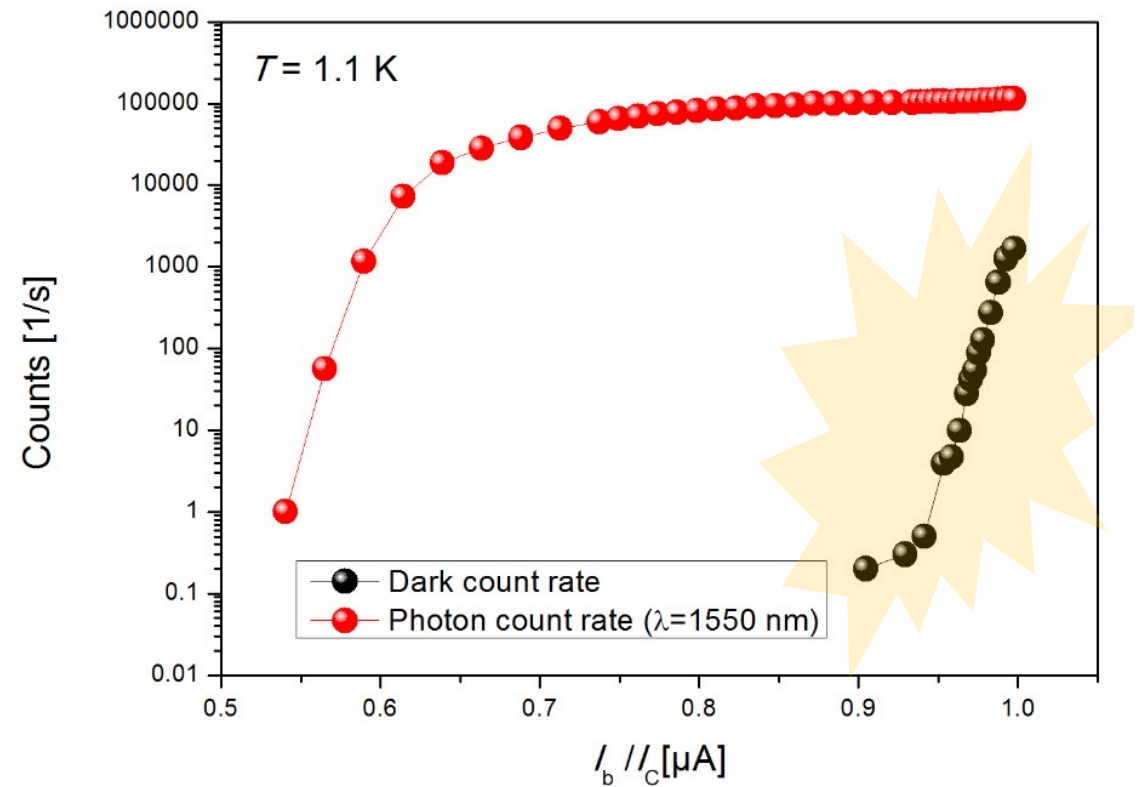
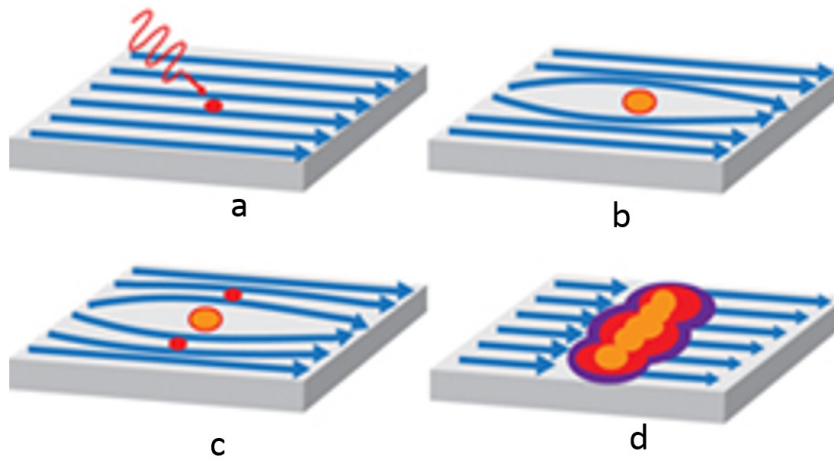
Idea: A. D. Semenov, G. N. Goltsman, A. A. Korneev (2001)



Photon counts & dark counts in SNSPDs



Photon counts & dark counts in SNSPDs



dark counts
=
thermal phase slips

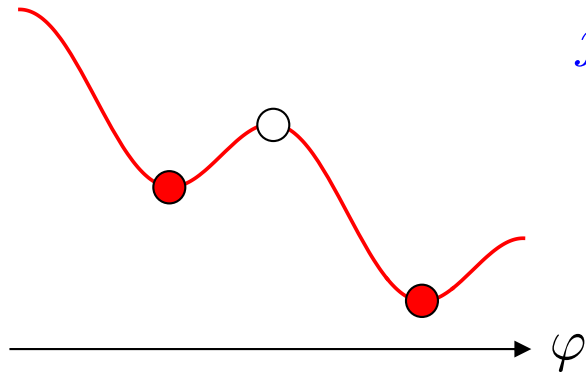
- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Thermal phase slips in a Josephson junction (0D)

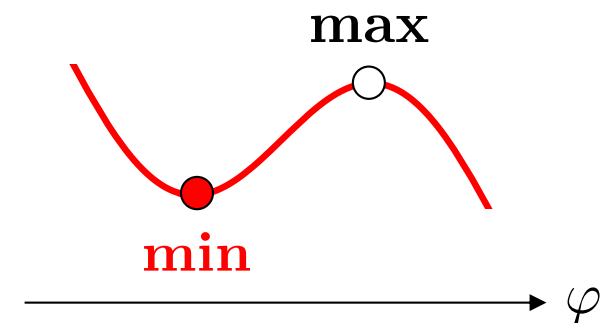
Current-biased JJ

$$U(\varphi) = E_J(1 - \cos \varphi) - E_J j \varphi$$

$$j = I/I_c$$



Activation rate: $e^{-\Delta F/T}$



$$\begin{aligned}\Delta F &= U(\varphi_{\max}) - U(\varphi_{\min}) \\ &= 2E_J \left[\sqrt{1 - j^2} - j \arccos j \right]\end{aligned}$$

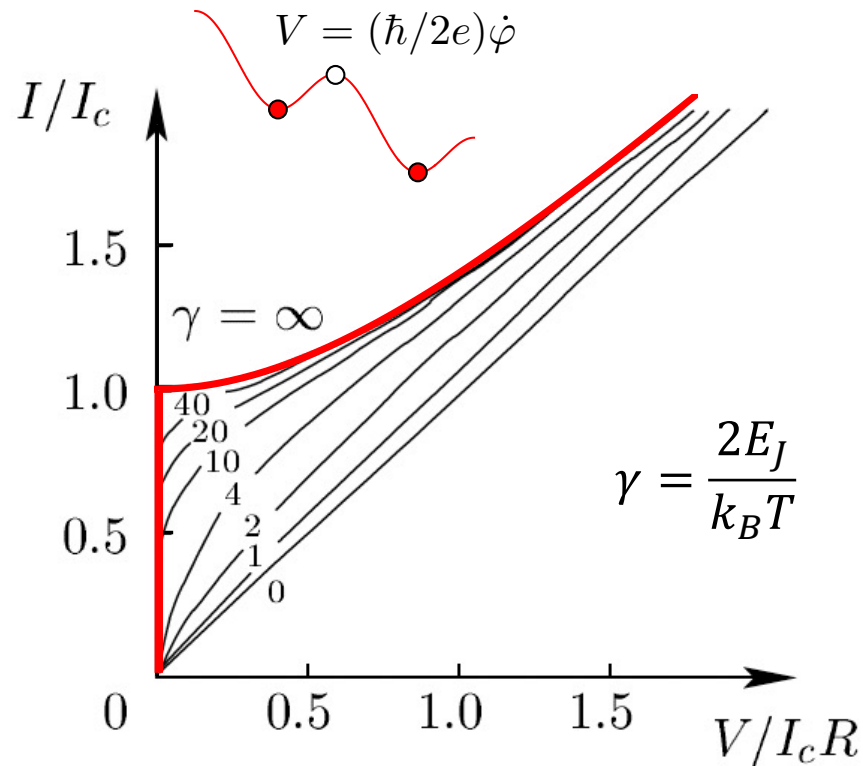
In the limit $I \rightarrow I_c$, the barrier has a power-law dependence:

$$\Delta F^{0D} \approx \frac{2^{5/2}}{3} E_J (1 - I/I_c)^{3/2}$$

Thermal phase slips: what's then?

- Zero-frequency measurements

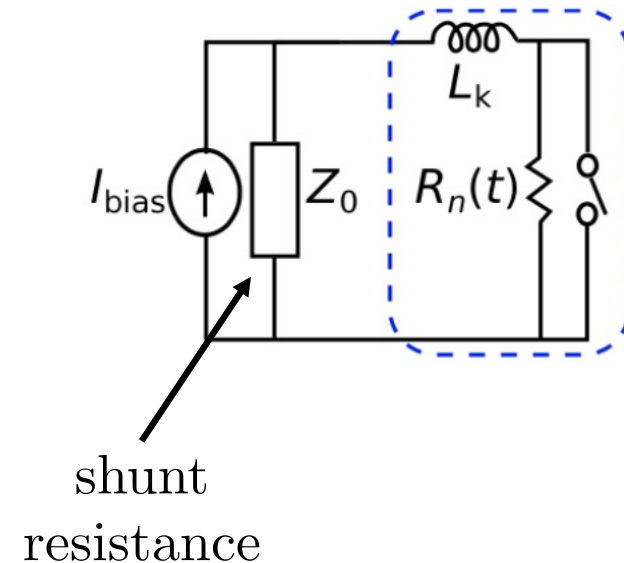
finite resistance



Wires: resistive state.....

- Time-resolved measurements

requires a mechanism to recover the initial superconducting state implemented in SNSPDs



resolving **individual** dark counts

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Supercurrent state in wires

Ginzburg-Landau free energy:

$$F = C \int dx (|\nabla \Delta|^2 - |\Delta|^2 + |\Delta|^4/2)$$

Ginzburg-Landau equation for $\Delta(\mathbf{r}) = |\Delta|e^{i\varphi}$:

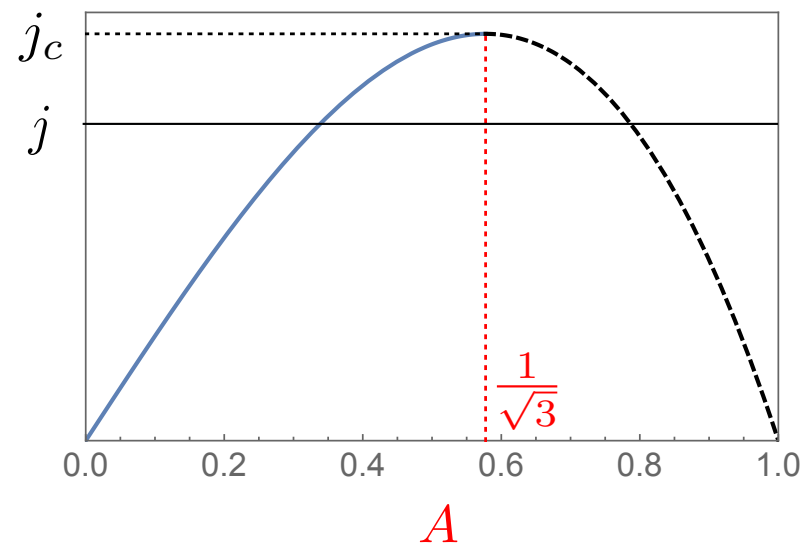
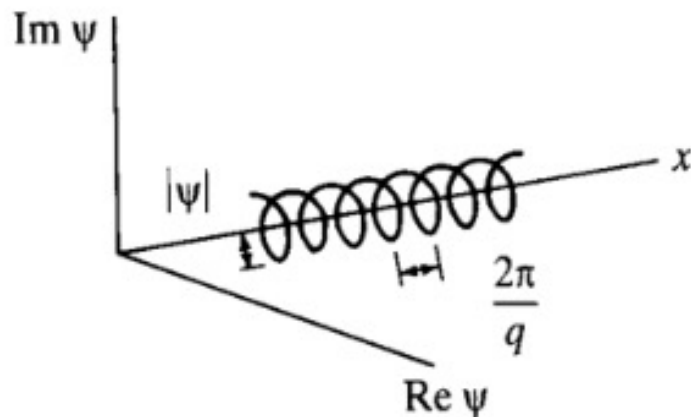
$$-\nabla^2 \Delta - \Delta + |\Delta|^2 \Delta = 0$$

Supercurrent:

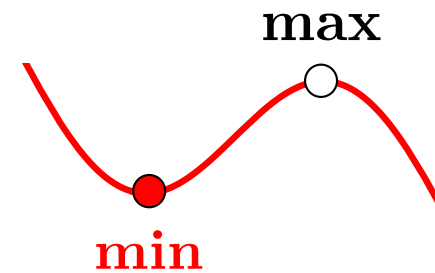
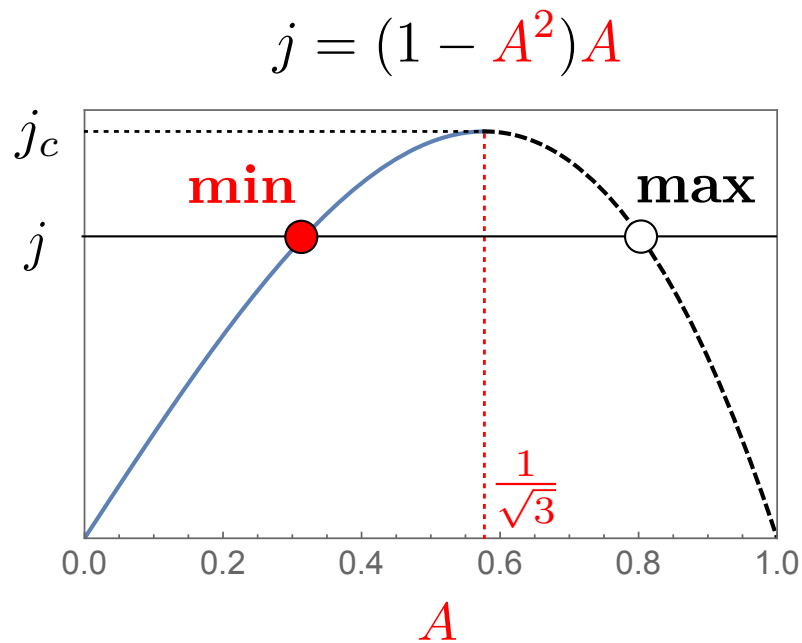
$$\mathbf{j} = \eta |\Delta|^2 \nabla \varphi$$

Supercurrent solution: $\Delta(x) = \sqrt{1 - A^2} e^{iAx}$

$$j = (1 - A^2)A$$



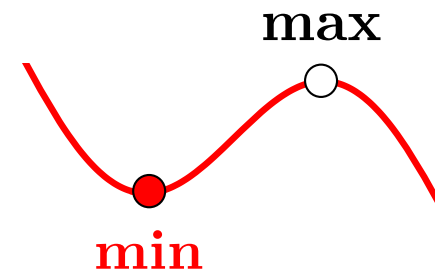
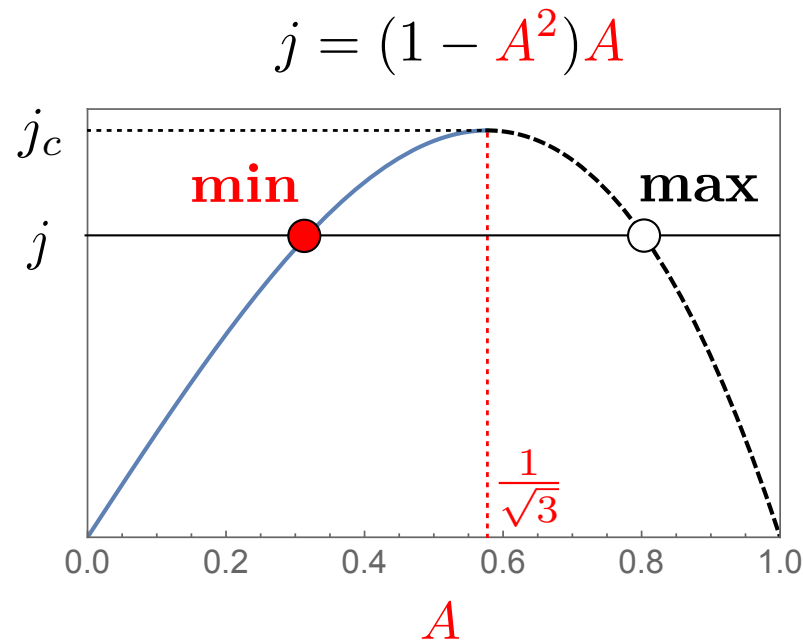
Phase slips: What is the barrier?



Naive barrier is extensive:

$$\Delta F \propto \int_0^L dx = L$$

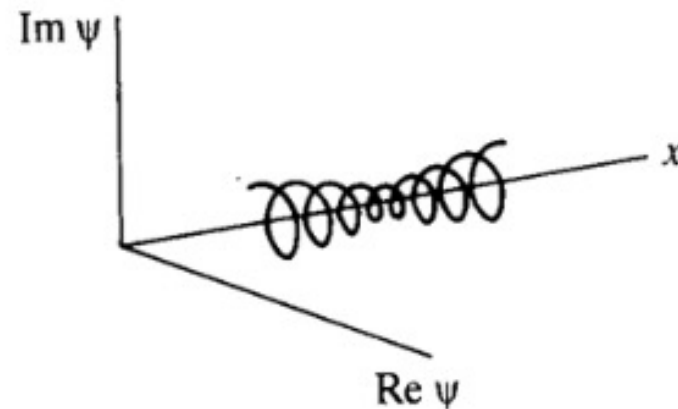
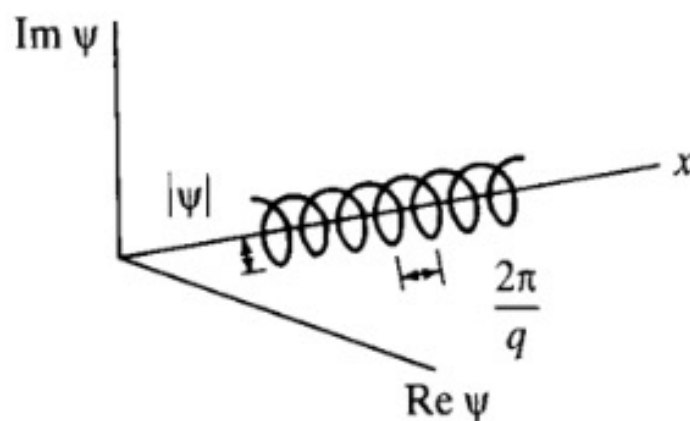
Phase slips: What is the barrier?



Naive barrier is extensive:

$$\Delta F \propto \int_0^L dx = L$$

The way out: a non-uniform solution [Langer and Ambegaokar (1967)]



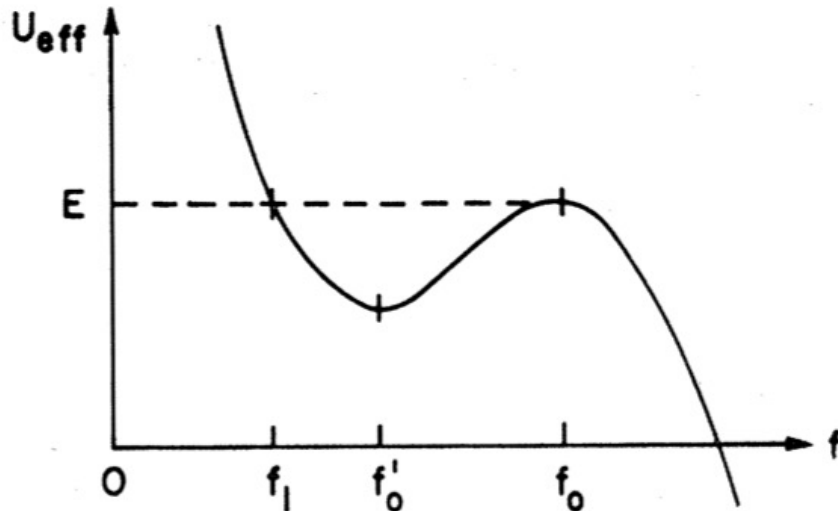
Langer-Ambegaokar instanton

Ginzburg-Landau equation (here $f = |\Delta|$):

$$-f'' - (1 - \varphi'^2)f + f^3 = 0, \quad (f^2\varphi')' = 0$$

Current conservation $j = f^2\varphi' = \text{const}$ allows to exclude the phase and to arrive at a mechanical problem

$$f'' = -\frac{dU_{\text{eff}}}{df}, \quad U_{\text{eff}}(f) = \frac{f^2}{2} + \frac{J^2}{2f^2} - \frac{f^4}{4}$$



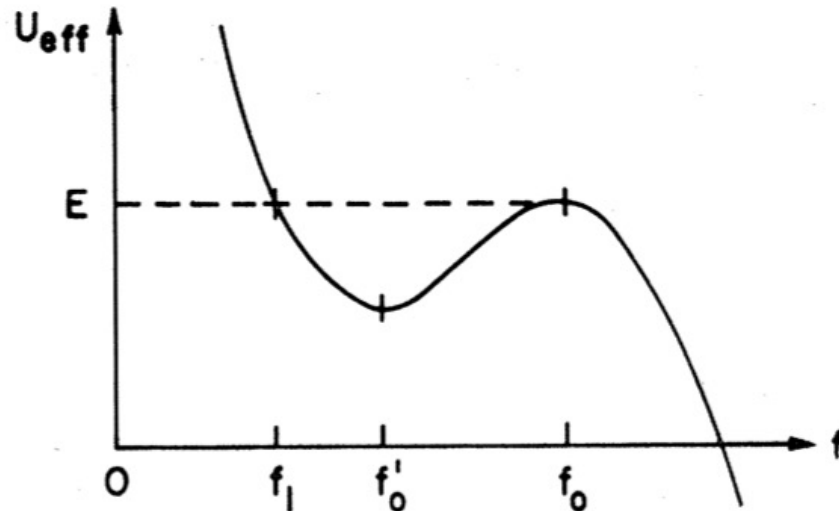
Langer-Ambegaokar instanton

Ginzburg-Landau equation (here $f = |\Delta|$):

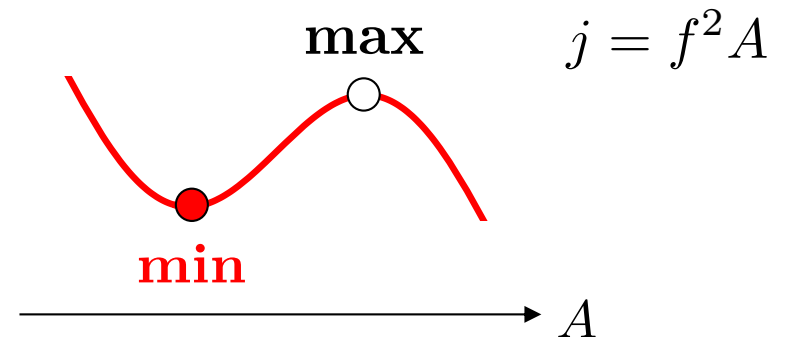
$$-f'' - (1 - \varphi'^2)f + f^3 = 0, \quad (f^2 \varphi')' = 0$$

Current conservation $j = f^2 \varphi' = \text{const}$ allows to exclude the phase and to arrive at a mechanical problem

$$f'' = -\frac{dU_{\text{eff}}}{df}, \quad U_{\text{eff}}(f) = \frac{f^2}{2} + \frac{J^2}{2f^2} - \frac{f^4}{4}$$



In the uniform case:



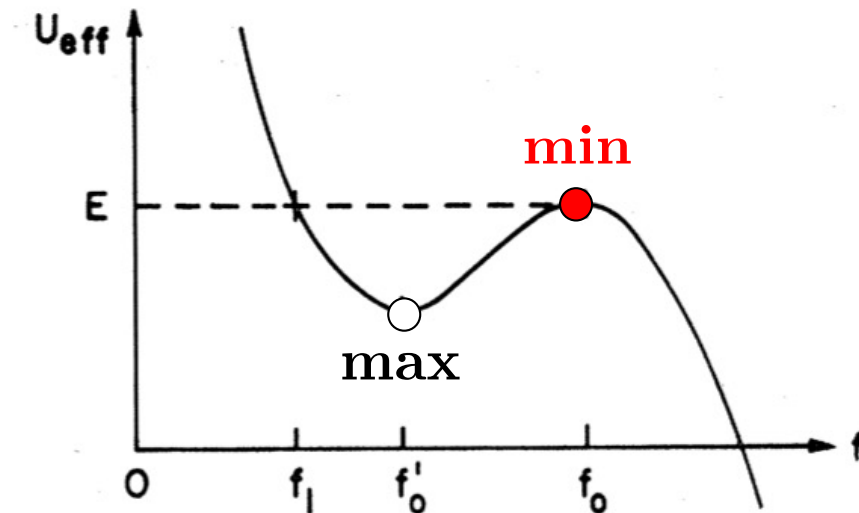
Langer-Ambegaokar instanton

Ginzburg-Landau equation (here $f = |\Delta|$):

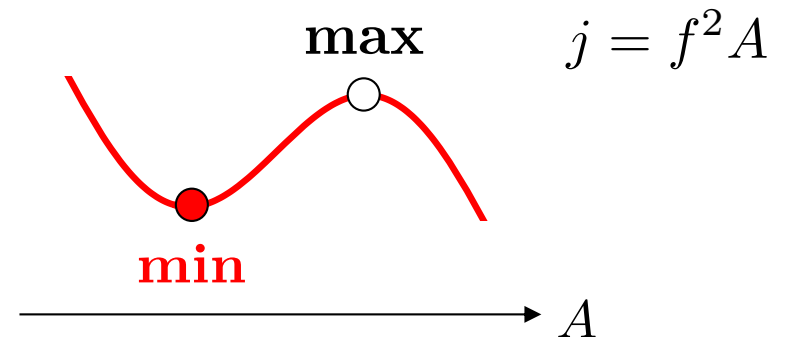
$$-f'' - (1 - \varphi'^2)f + f^3 = 0, \quad (f^2 \varphi')' = 0$$

Current conservation $j = f^2 \varphi' = \text{const}$ allows to exclude the phase and to arrive at a mechanical problem

$$f'' = -\frac{dU_{\text{eff}}}{df}, \quad U_{\text{eff}}(f) = \frac{f^2}{2} + \frac{J^2}{2f^2} - \frac{f^4}{4}$$



In the uniform case:



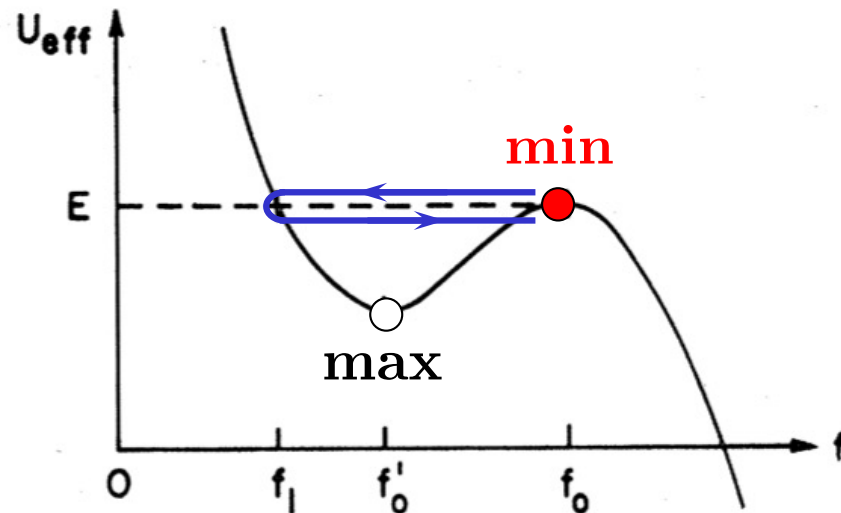
Langer-Ambegaokar instanton

Ginzburg-Landau equation (here $f = |\Delta|$):

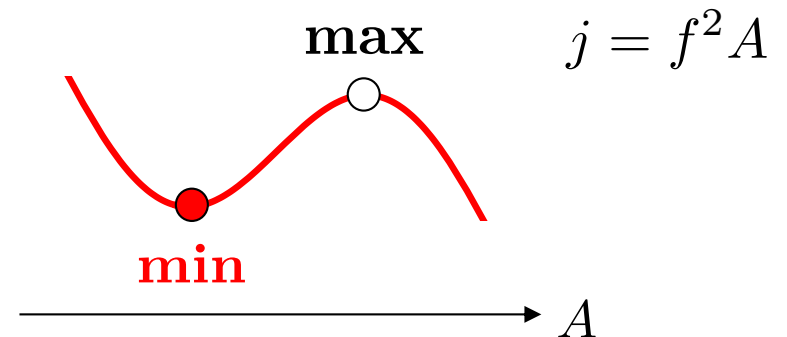
$$-f'' - (1 - \varphi'^2)f + f^3 = 0, \quad (f^2 \varphi')' = 0$$

Current conservation $j = f^2 \varphi' = \text{const}$ allows to exclude the phase and to arrive at a mechanical problem

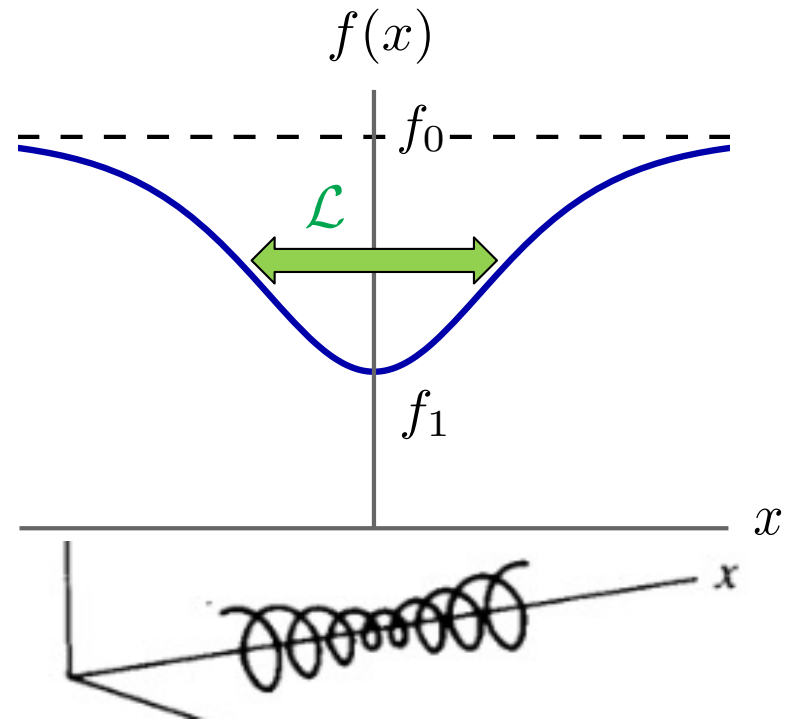
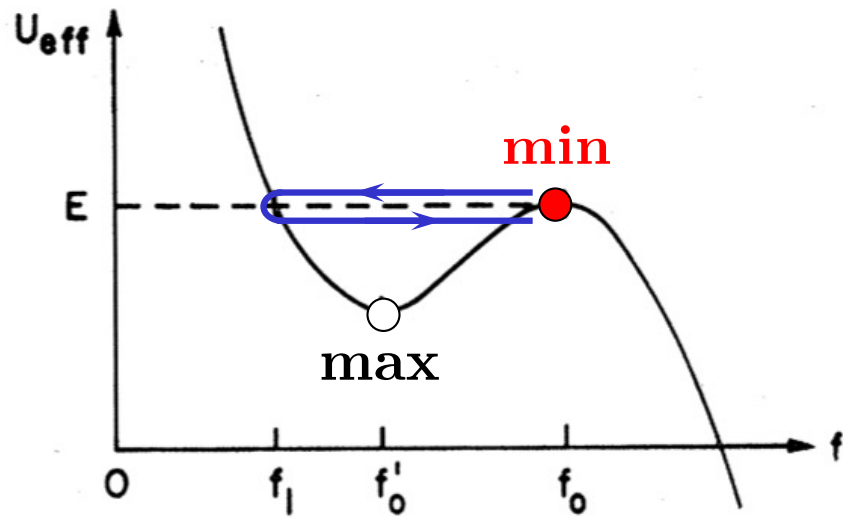
$$f'' = -\frac{dU_{\text{eff}}}{df}, \quad U_{\text{eff}}(f) = \frac{f^2}{2} + \frac{J^2}{2f^2} - \frac{f^4}{4}$$



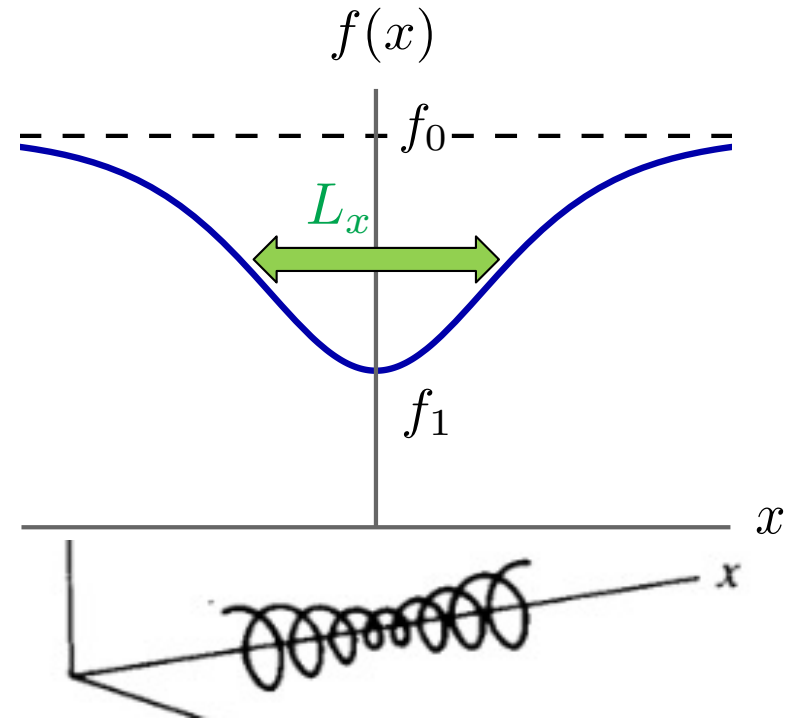
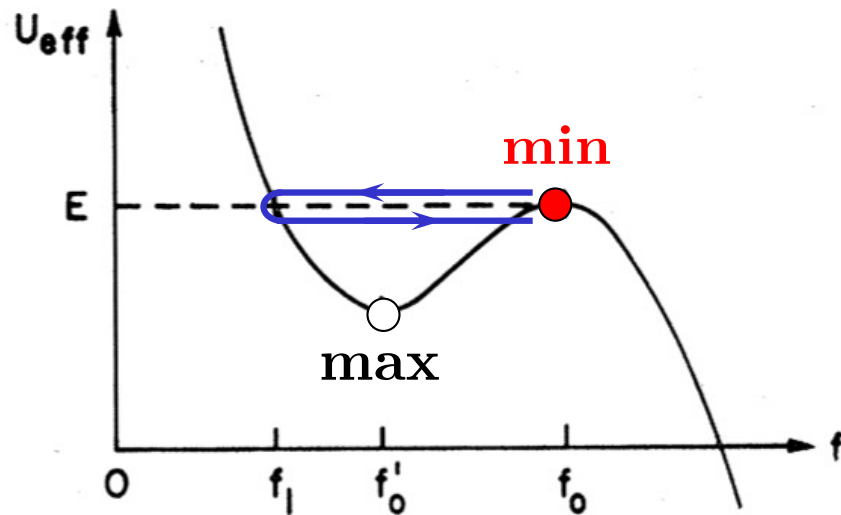
In the uniform case:



Langer-Ambegaokar instanton



Langer-Ambegaokar instanton



Analytic solution:

$$f(x) = f_0 - (f_0 - f_1) / \cosh^2(x/L_x)$$

Instanton size:

$$L_x \sim \xi(T)(1 - I/I_c)^{-1/4}$$

Energy barrier:

$$\begin{aligned} \Delta F^{1D} &\propto (1 - I/I_c)^{3/2} L_x \\ &\propto (1 - I/I_c)^{5/4} \end{aligned}$$

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

GL instanton in 2D

Ginzburg-Landau equation for the complex $\Delta(\mathbf{r})$:

$$-\nabla^2 \Delta - \Delta + |\Delta|^2 \Delta = 0$$

Boundary condition at infinity: $\mathbf{j} = |\Delta|^2 \nabla \varphi \rightarrow j_0 \mathbf{e}_x$

Current conservation: $\text{div } \mathbf{j} = \nabla(|\Delta|^2 \nabla \varphi) = 0$

- 1D: $|\Delta|^2 \varphi' = j_0 = \text{const}$

closed equation for $|\Delta(x)|$
[Langer, Ambegaokar (1964)]

- 2D: nonlocality

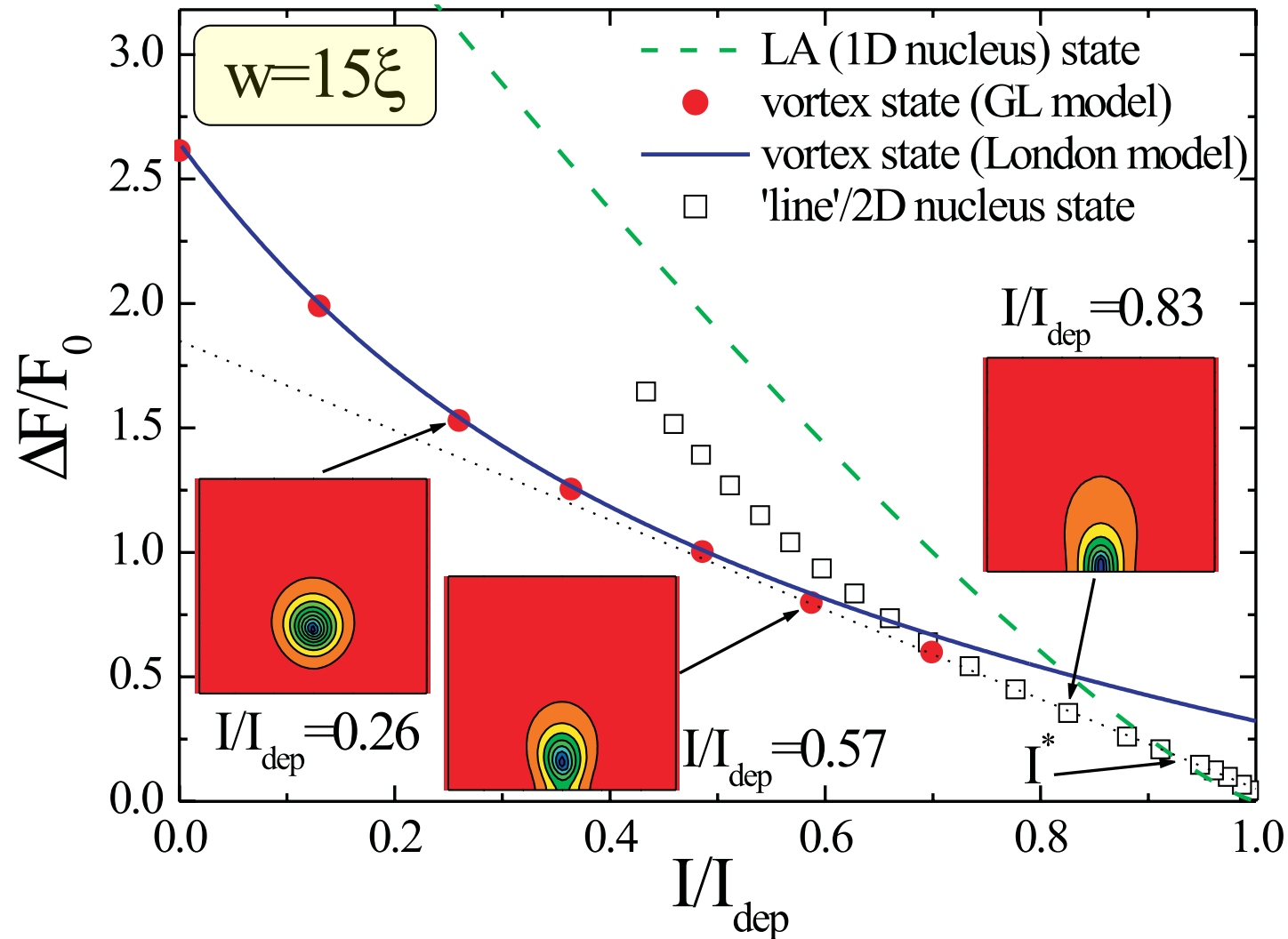
coupled PDEs
for $|\Delta(x, y)|$ and $\varphi(x, y)$

No way?

GL instanton in 2D strips: numerics

L. N. Bulaevskii, M. J. Graf, C. D. Batista, V. G. Kogan (2011)

D. Y. Vodolazov (2012)



- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

GL instanton in 2D

Ginzburg-Landau equation for the complex $\Delta(\mathbf{r})$:

$$-\nabla^2 \Delta - \Delta + |\Delta|^2 \Delta = 0$$

Boundary condition at infinity: $\mathbf{j} = |\Delta|^2 \nabla \varphi \rightarrow j_0 \mathbf{e}_x$

Current conservation: $\text{div } \mathbf{j} = \nabla(|\Delta|^2 \nabla \varphi) = 0$

- 1D: $|\Delta|^2 \varphi' = j_0 = \text{const}$

closed equation for $|\Delta(x)|$
[Langer, Ambegaokar (1967)]

- 2D: **nonlocality**

coupled PDEs
for $|\Delta(x, y)|$ and $\varphi(x, y)$

No way?

Stream function

Description of an incompressible flow by the stream function $\psi(\mathbf{r})$

$$\mathbf{j} = (\psi_y, -\psi_x)$$

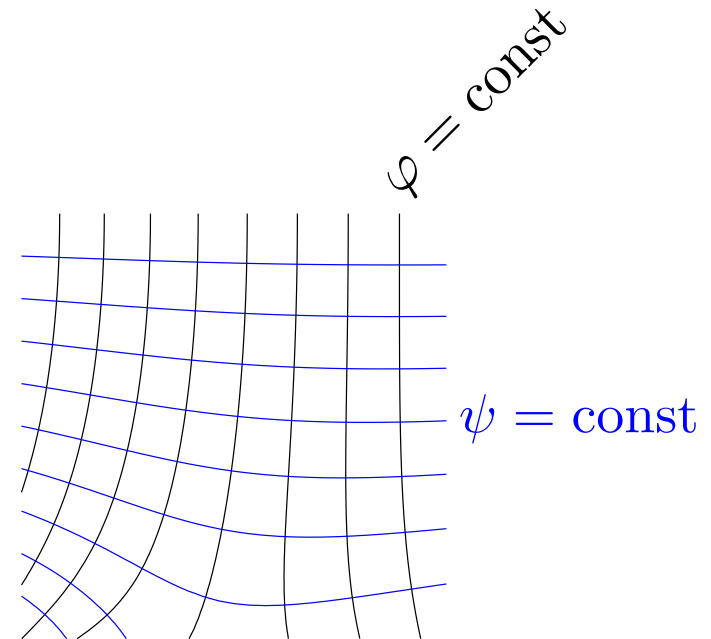
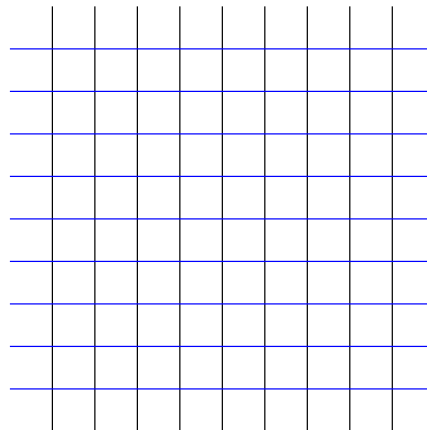
- Automatically resolves $\text{div } \mathbf{j} = 0$
- How to get $|\Delta(\mathbf{r})|$ and $\varphi(\mathbf{r})$?

$$\nabla\psi\nabla\varphi = 0$$

That is, ψ and φ are orthogonal curvilinear coordinates on the plane

Uniform current:

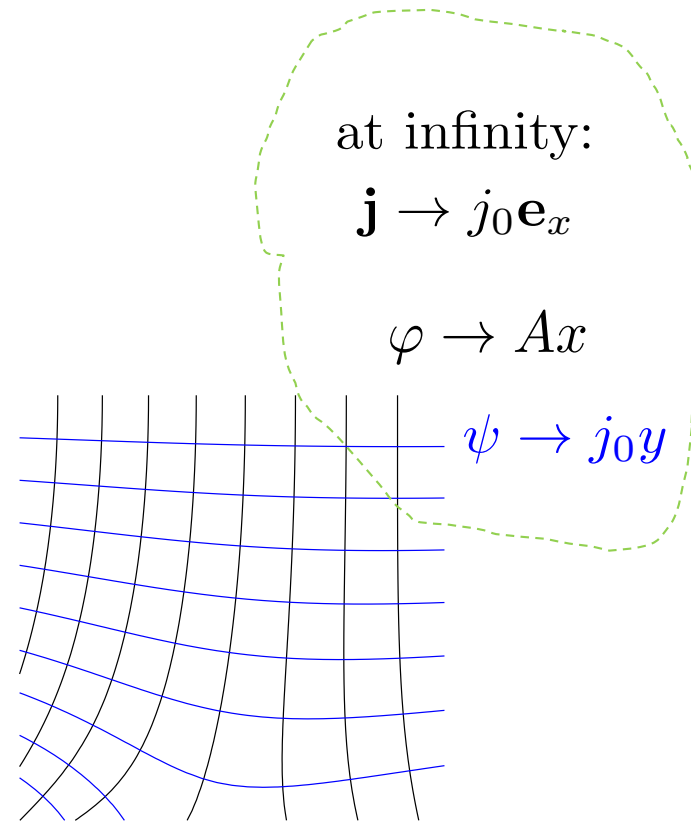
$$\psi = j_0 y \text{ and } \varphi = Ax$$



From stream function to order parameter

1. Determine the **phase** φ
by solving $\nabla\psi\nabla\varphi = 0$
(**nonlocal**)

Solution for $\phi(\mathbf{r})$
exists and is unique
provided $\nabla\psi \neq 0$
(no vortices)



2. Determine the **modulus** $|\Delta|$ from $\mathbf{j} = |\Delta|^2 \nabla\varphi$ (local)

Result: **nonlocal** (numerical) and nonlinear $F[\psi]$

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Near the critical current

At $I \rightarrow I_c$, instanton is a **weak perturbation** of the uniform solution

- Stream function: $\psi = j_0(y + f_y)$, and f is **small**
- Phase: $\varphi = A(x + g)$ $g = g^{(1)} + g^{(2)} + \dots$, where $g^{(n)} \propto f^n$
- Modulus: $|\Delta|^2 = \Delta_0^2[1 + (1 + f_{yy})/(1 + g_x)]$

Near the critical current

At $I \rightarrow I_c$, instanton is a **weak perturbation** of the uniform solution

- Stream function: $\psi = j_0(y + f_y)$, and f is **small**
- Phase: $\varphi = A(x + g)$ $g = g^{(1)} + g^{(2)} + \dots$, where $g^{(n)} \propto f^n$
- Modulus: $|\Delta|^2 = \Delta_0^2[1 + (1 + f_{yy})/(1 + g_x)]$

$$\nabla\psi\nabla\varphi = 0 \quad \Rightarrow \quad f_{xy}(1 + g_x) + (1 + f_{yy})g_y = 0$$

Near the critical current

At $I \rightarrow I_c$, instanton is a **weak perturbation** of the uniform solution

- Stream function: $\psi = j_0(y + f_y)$, and f is **small**
- Phase: $\varphi = A(x + g)$ $g = g^{(1)} + g^{(2)} + \dots$, where $g^{(n)} \propto f^n$
- Modulus: $|\Delta|^2 = \Delta_0^2[1 + (1 + f_{yy})/(1 + g_x)]$

$$\nabla\psi\nabla\varphi = 0 \quad \Rightarrow \quad f_{xy}(1 + g_x) + (1 + f_{yy})g_y = 0$$

- 1 order: $g_y^{(1)} = -f_{xy} \longrightarrow g^{(1)} = -f_x$ (local!)
 - 2 order: $g_y^{(2)} = (f_{xx} + f_{yy})f_{xy}$
 - 3 order: $g_y^{(3)} = -(f_{xx} + f_{yy})f_{xy}f_{yy} - g_x^{(2)}f_{xy}$
- } (nonlocal)

Near the critical current

$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local $\varepsilon f_{xx}^2, f_{xy}^2, f_{yy}^2, f_{xxx}^2, f_{xxy}^2, f_{xyy}^2, f_{yyy}^2$
- Cubic terms: plenty of local and nonlocal terms

Near the critical current

$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local $\varepsilon f_{xx}^2, f_{xy}^2, f_{yy}^2, f_{xxx}^2, f_{xxy}^2, f_{xyy}^2, f_{yyy}^2$
- Cubic terms: plenty of local and nonlocal terms

Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}} \quad L_y \sim \frac{1}{\varepsilon} \quad L_y \gg L_x$

Near the critical current

$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local $\varepsilon f_{xx}^2, f_{xy}^2, f_{yy}^2, f_{xxx}^2, f_{xxy}^2, f_{xyy}^2, f_{yyy}^2$
- Cubic terms: plenty of local and nonlocal terms

Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}} \quad L_y \sim \frac{1}{\varepsilon} \quad L_y \gg L_x$

Near the critical current

$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local εf_{xx}^2 , f_{xy}^2 , f_{yy}^2 , f_{xxx}^2 , f_{xxy}^2 , f_{xyy}^2 , f_{yyy}^2
- Cubic terms: plenty of local and nonlocal terms

Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}}$ $L_y \sim \frac{1}{\varepsilon}$ $L_y \gg L_x$

$$F = C \int dx dy \left(\frac{\varepsilon}{3} f_{xx}^2 + \frac{2}{9} f_{xy}^2 + \frac{1}{6} f_{xxx}^2 + \frac{2}{9} f_{xx}^3 \right)$$

Near the critical current

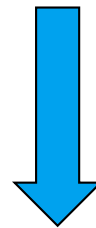
$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local εf_{xx}^2 , f_{xy}^2 , ~~f_{yy}^2~~ , ~~f_{xxx}^2~~ , ~~f_{xxy}^2~~ , ~~f_{xyy}^2~~ , ~~f_{yyy}^2~~
- Cubic terms: ~~plenty~~ of local and ~~nonlocal~~ terms

Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}}$ $L_y \sim \frac{1}{\varepsilon}$ $L_y \gg L_x$

$$F = C \int dx dy \left(\frac{\varepsilon}{3} f_{xx}^2 + \frac{2}{9} f_{xy}^2 + \frac{1}{6} f_{xxx}^2 + \frac{2}{9} f_{xx}^3 \right)$$

Rescaling



$$F = 2^{1/2} 3^{-3/2} \varepsilon^{3/2} SC$$

$$S = \int d\bar{x} d\bar{y} \left(\frac{1}{2} \bar{f}_{\bar{x}\bar{x}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{y}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{x}\bar{x}}^2 + \frac{1}{3} \bar{f}_{\bar{x}\bar{x}}^3 \right) \quad f = \bar{f}/2$$

Boussinesq equation

$$S = \int d\bar{x} d\bar{y} \left(\frac{1}{2} \bar{f}_{\bar{x}\bar{x}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{y}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{x}\bar{x}}^2 + \frac{1}{3} \bar{f}_{\bar{x}\bar{x}}^3 \right)$$

Saddle-point equation written in terms of $u = \bar{f}_{\bar{x}\bar{x}}$:

$$u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} - u_{\bar{x}\bar{x}\bar{x}\bar{x}} + (u^2)_{\bar{x}\bar{x}} = 0$$

Boussinesq equation in the **elliptic** form

In its original (**hyperbolic**) formulation, with $\bar{y} = it$ and t being time, this equation describes propagation of shallow water waves in the long-wavelength limit [**J. Boussinesq (1872)**].

- Nonlinear PDE **integrable** by the inverse scattering method
[V. E. Zakharov, S. V. Manakov, S. P. Novicov, and L. P. Pitaevsky, *Theory of Solitons: The Inverse Scattering Method* (1984)]
- Existence of solitary waves (**solitons**)

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Instanton in the Boussinesq equation

$$u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} - u_{\bar{x}\bar{x}\bar{x}\bar{x}} + (u^2)_{\bar{x}\bar{x}} = 0$$

We need a nontrivial localized solution vanishing at $r \rightarrow \infty$

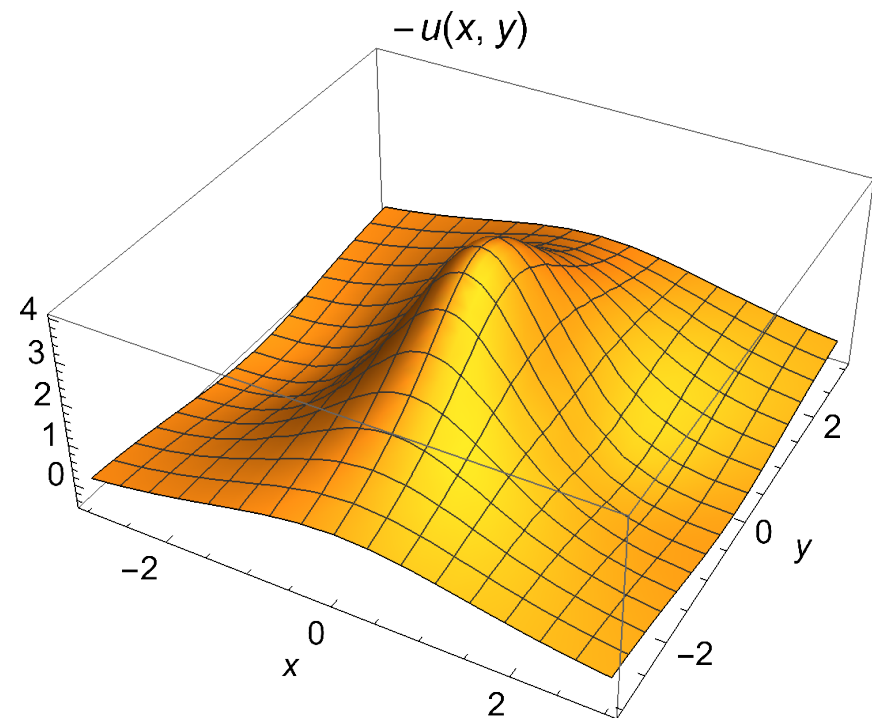
It can be found in the form inspired by Hirota:

$$u_B = -6\partial_{\bar{x}}^2 \ln(\bar{x}^2 + \bar{y}^2 + 3)$$

$$\bar{f}_B = -6 \ln(\bar{x}^2 + \bar{y}^2 + 3)$$

Instanton action

$$S = 8\pi$$



Boussinesq instanton: variational approach

$$S = \int d\bar{x} d\bar{y} \left(\frac{1}{2} \bar{f}_{\bar{x}\bar{x}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{x}\bar{x}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{y}}^2 + \frac{1}{3} \bar{f}_{\bar{x}\bar{x}}^3 \right)$$

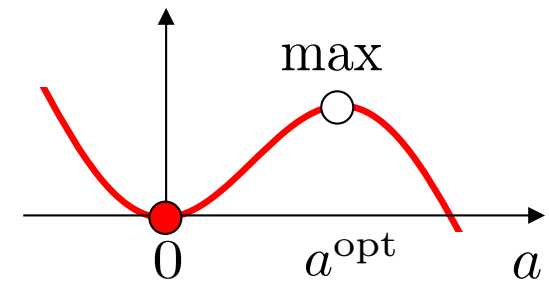
One can look for the saddle point of S using the ansatz $\bar{f} = az(\bar{x}, \bar{y})$, where the spatial profile $z(\bar{x}, \bar{y}) = \zeta(c_1\bar{x}, c_2\bar{y})$ determined by a probe function $\zeta(\cdot, \cdot)$ with $\zeta(0, 0) = 1$. Optimization parameters: a and $c_{1,2}$.

- First we maximize with respect to the amplitude:

$$a^{\text{opt}} = - \frac{\langle z_{\bar{x}\bar{x}}^2 + z_{\bar{x}\bar{x}\bar{x}}^2 + z_{\bar{x}\bar{y}}^2 \rangle}{\langle z_{\bar{x}\bar{x}}^3 \rangle} > 0 \quad \text{where } \langle \dots \rangle = \int d\bar{x} d\bar{y} (\dots)$$

- Then we minimize with respect to $c_{1,2}$:

$$S^{\text{opt}}[z] = \min_{c_1, c_2} \frac{\langle z_{\bar{x}\bar{x}}^2 + z_{\bar{x}\bar{x}\bar{x}}^2 + z_{\bar{x}\bar{y}}^2 \rangle^3}{6 \langle z_{\bar{x}\bar{x}}^3 \rangle^2}$$



Boussinesq instanton: variational approach

Ansatz for $\bar{f}(\bar{x}, \bar{y})$	c_1^{opt}	c_2^{opt}	$S^{\text{opt}}/8\pi$
$-6 \ln(\bar{x}^2 + \bar{y}^2 + 3)$			1
$a/(1 + c_1^2 \bar{x}^2 + c_2^2 \bar{y}^2)^{1/2}$	0.4	0.4	1.120
$a/(1 + c_1^2 \bar{x}^2 + c_2^2 \bar{y}^2)$	0.322	0.322	1.286
$a/(1 + c_1^2 \bar{x}^2)(1 + c_2^2 \bar{y}^2)$	0.298	0.422	1.612
$a/\cosh(c_1 \bar{x}) \cosh(c_2 \bar{y})$	0.459	0.543	1.616
$a \exp(-c_1^2 \bar{x}^2 - c_2^2 \bar{y}^2)$	0.365	0.365	3.003

- $S^{\text{opt}} > 8\pi$ for all probe functions
- power-law functions with a small exponent provide a better approximation
- the best candidates depend on $\bar{x}^2 + \bar{y}^2$

All these facts confirm $\bar{f}_B = -6 \ln(\bar{x}^2 + \bar{y}^2 + 3)$ as the saddle-point solution with the minimal energy

Free energy barrier

$$\Delta F^{2D} = c_2 \varepsilon_{\text{cond}} d\xi^2 (1 - I/I_c)^{3/4}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

Free energy barrier

$$\Delta F^{2\text{D}} = c_2 \varepsilon_{\text{cond}} d \xi^2 (1 - I/I_c)^{3/4}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

$$\Delta F^{1\text{D}} = c_1 \varepsilon_{\text{cond}} d w \xi (1 - I/I_c)^{5/4}$$

$$\Delta F^{0\text{D}} = c_0 \varepsilon_{\text{cond}} d w L (1 - I/I_c)^{3/2}$$

Free energy barrier

$$\Delta F^{2D} = c_2 \varepsilon_{\text{cond}} d \xi^2 (1 - I/I_c)^{3/4}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

$$\sim \frac{L_x}{L} \frac{L_y}{w} \Delta F^{0D} \quad \frac{6}{4} - \frac{1}{4} - \frac{1}{2}$$

$$\Delta F^{1D} = c_1 \varepsilon_{\text{cond}} d w \xi (1 - I/I_c)^{5/4}$$

$$\sim \frac{L_x}{L} \Delta F^{0D} \quad \frac{6}{4} - \frac{1}{4}$$

$$\Delta F^{0D} = c_0 \varepsilon_{\text{cond}} d w L (1 - I/I_c)^{3/2}$$

$$\frac{6}{4}$$

Length scales:

$$L_x \sim \frac{\xi}{\sqrt{\varepsilon}} \sim \frac{\xi}{(1 - I/I_c)^{1/4}}$$

$$L_y \sim \frac{\xi}{\varepsilon} \sim \frac{\xi}{(1 - I/I_c)^{1/2}}$$

Supercurrent pattern at the saddle point

Supercurrent flow pattern at f_B :

$$\frac{j_x}{j_0} = 1 - \frac{18\varepsilon^2(2\varepsilon x^2 - 3\varepsilon^2 y^2 + 3)}{(2\varepsilon x^2 + 3\varepsilon^2 y^2 + 3)^2}$$

$$\frac{j_y}{j_0} = -\frac{72\varepsilon^3 xy}{(2\varepsilon x^2 + 3\varepsilon^2 y^2 + 3)^2}$$

Current at the instanton center:

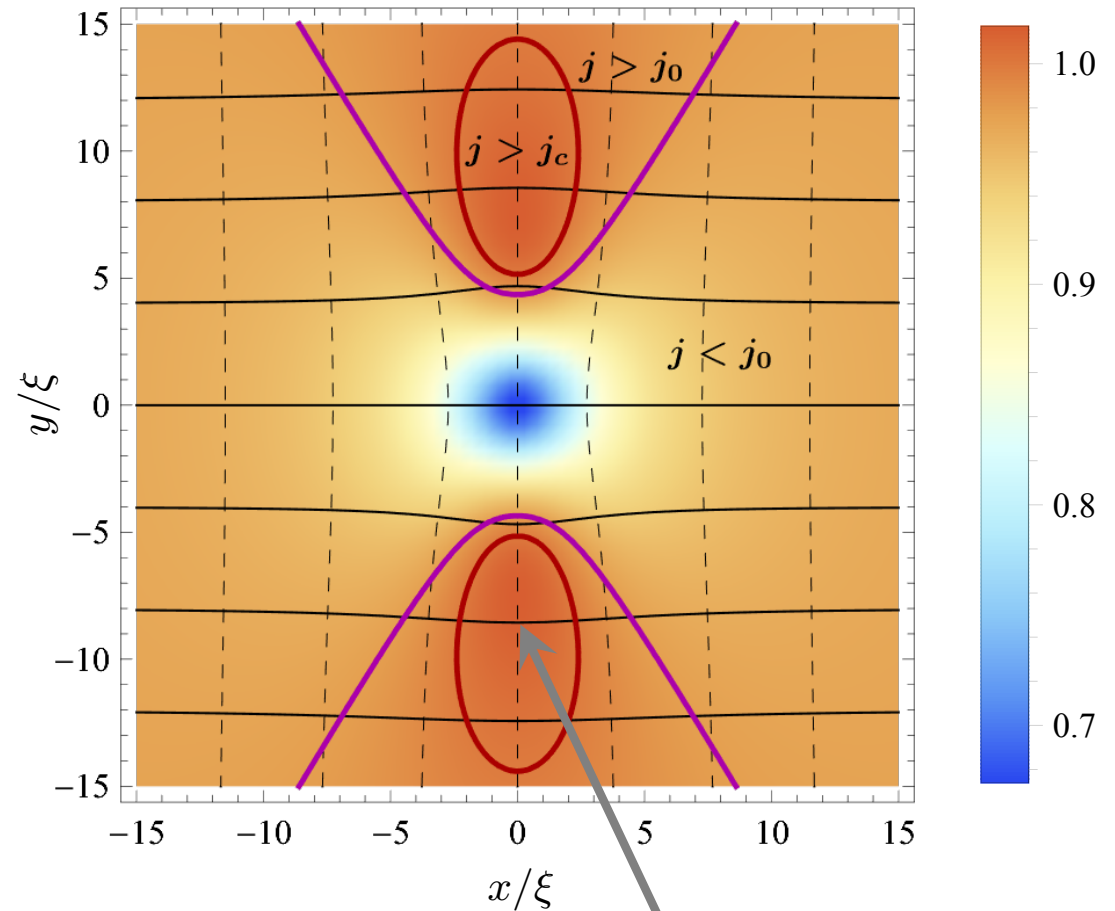
$$\frac{j_B(0)}{j_0} = 1 - 6\varepsilon^2$$

formally vanishes at $I = 0.926I_c$

$$\varepsilon = 1 - 3A^2$$

$$j = (1 - A^2)A$$

↑
narrow region of applicability



$$I = 0.98I_c$$

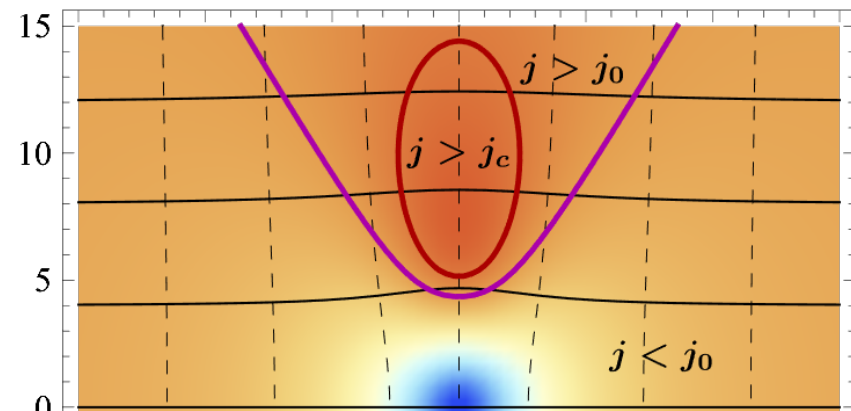
$$I_{\max} = 1.02I_c$$

Thermal phase slips in wide strips

- Wide strips $w \gg L_y$

Instanton is formed near the edge

$$\Delta F = \frac{1}{2} \Delta F^{2D}$$



- Not so wide strips $w \ll L_y$

1D LA instanton

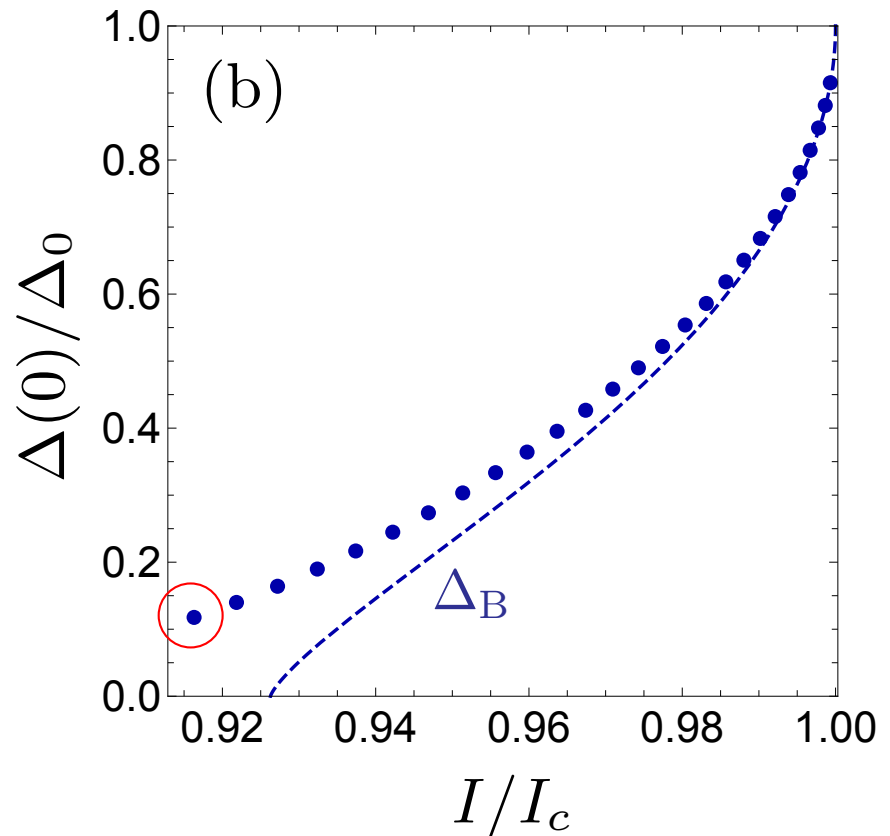
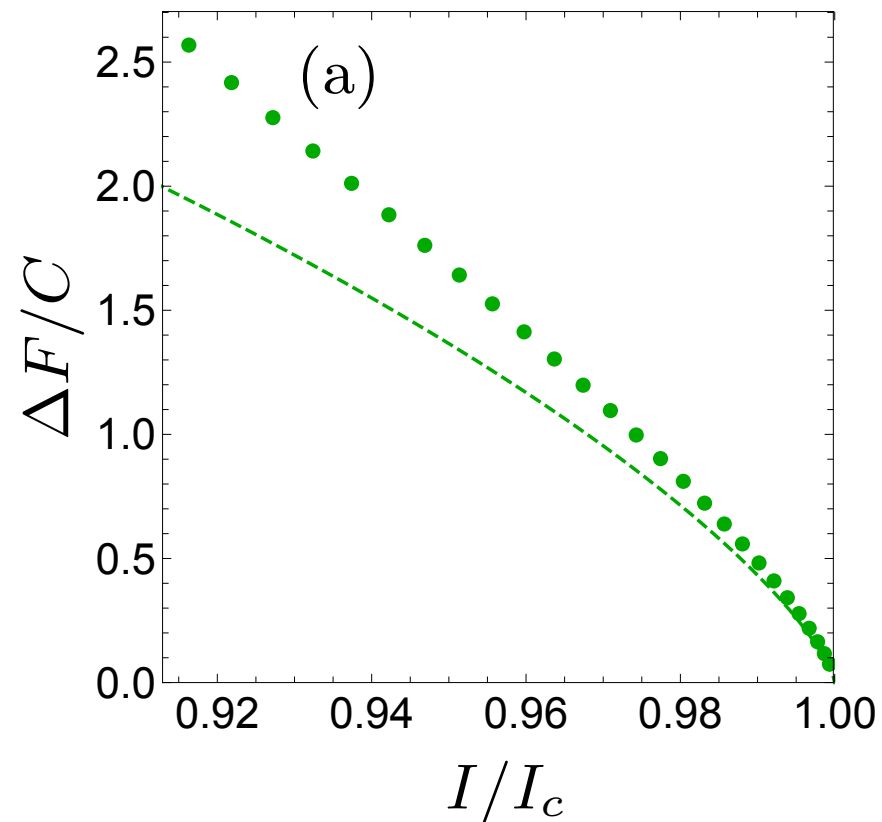
$$\Delta F = \Delta F^{1D}$$

- Superconducting photon detectors
- Thermal phase slips
 - 0D: Josephson junction
 - 1D: Langer-Ambegaokar theory
 - 2D: Status quo
- Analytical solution in 2D at $I \rightarrow I_c$
 - Stream function
 - Boussinesq equation
 - Its instanton solution
- Away from I_c & topological transition between instantons

Variational approach at finite ε

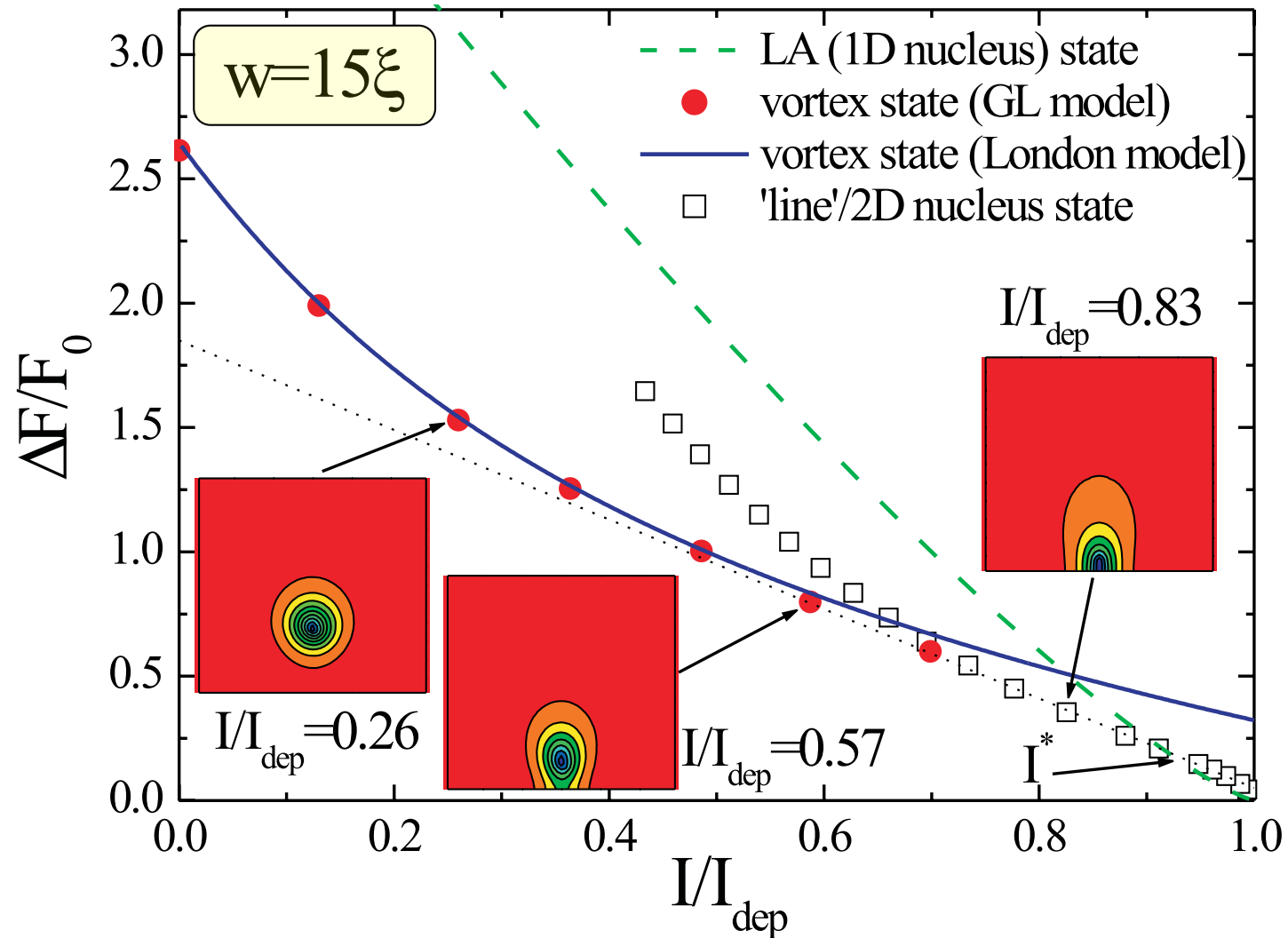
Boussinesq instanton: $f_B(x, y) = -3 \ln(1 + 2\varepsilon x^2/3 + \varepsilon^2 y^2)$ ($I \rightarrow I_c$)

Boussinesq-like ansatz: $f(x, y) = -a \ln(1 + c_1^2 x^2 + c_2^2 y^2)$



GL instanton in 2D strips: numerics

D. Y. Vodolazov (2012)

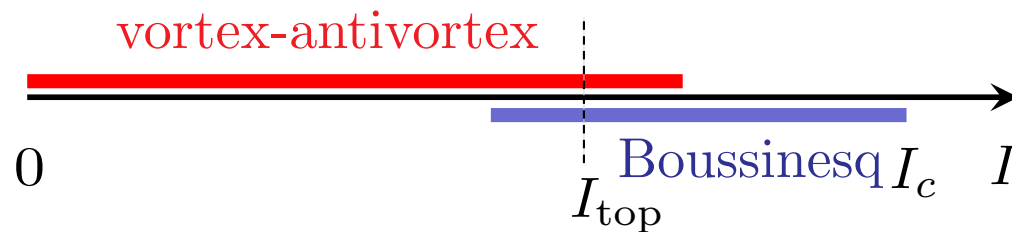


Topological transition between instantons

- **First order scenario**

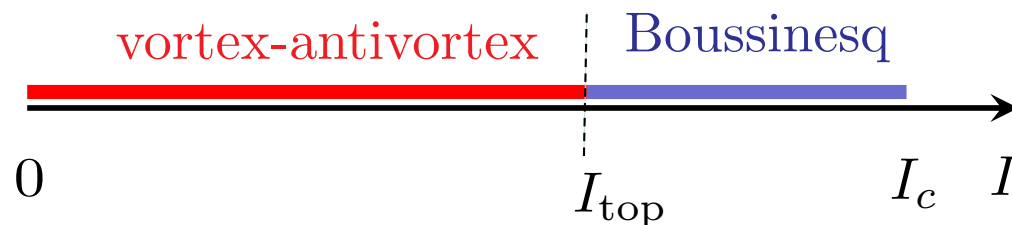
VA and B instantons coexist in some region of I

Solution with the lowest ΔF wins



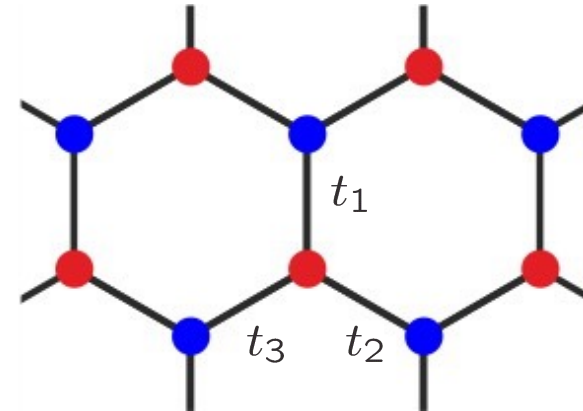
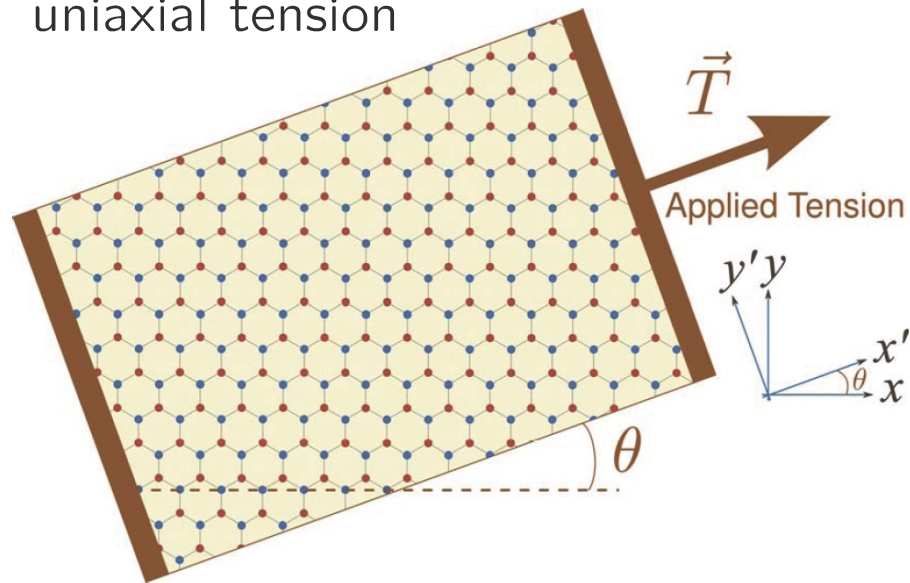
- **Second order scenario**

VA and B instantons continuously transform to each other



Topological transition in stretched graphene

Graphene under uniaxial tension



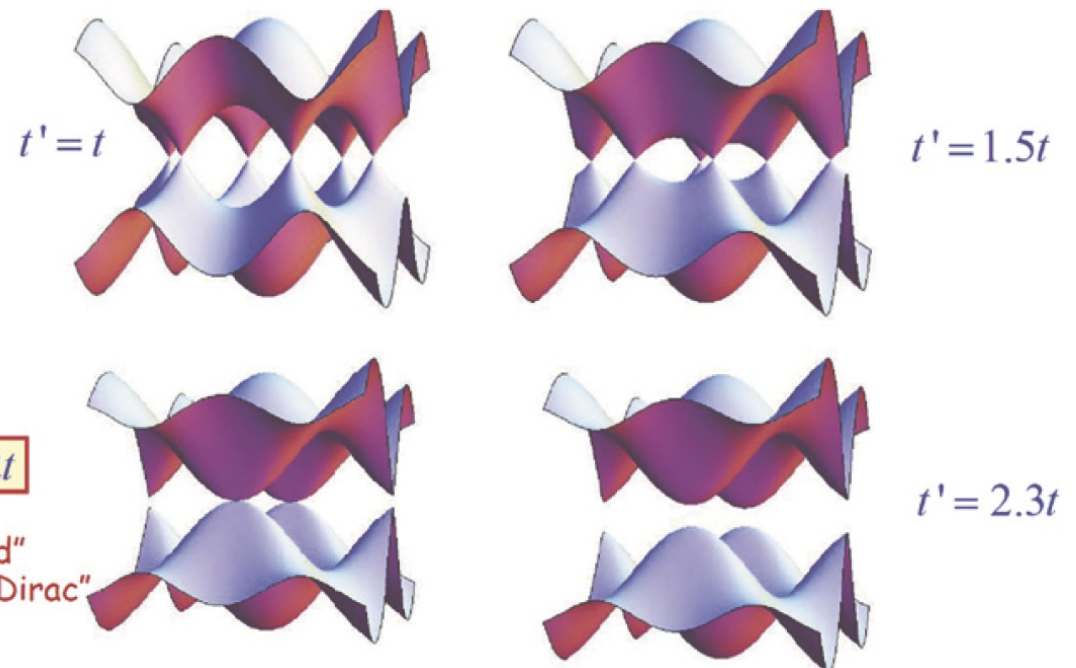
Toy model:

$$t_1 = t', \quad t_2 = t_3 = t$$

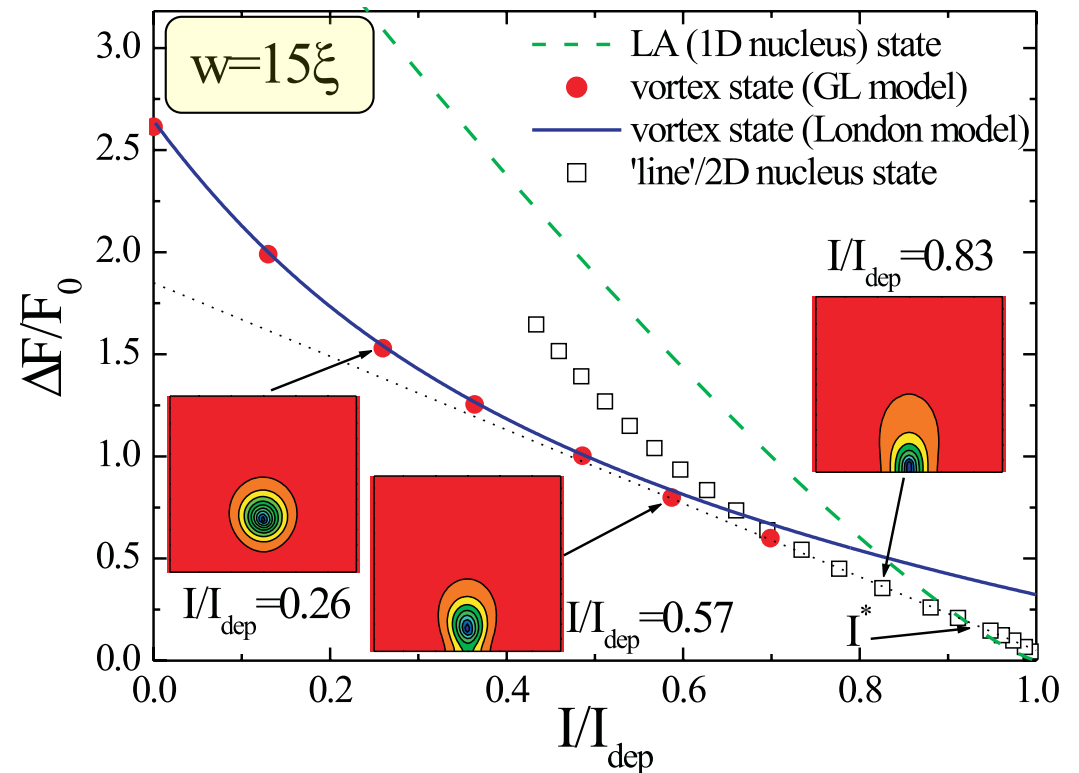
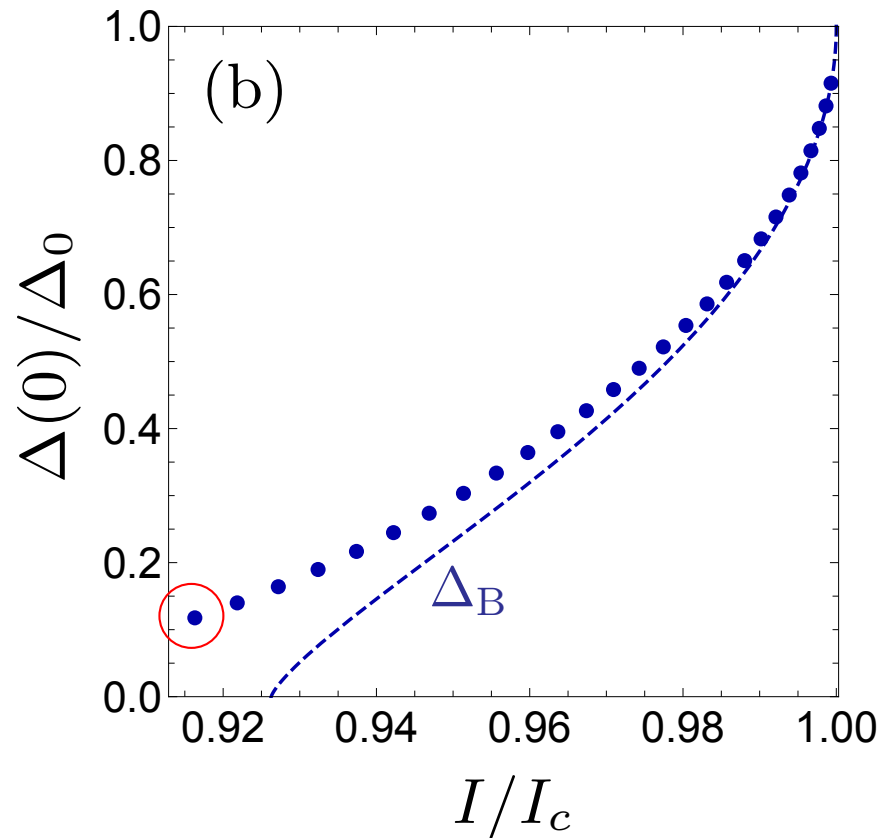
$$E(\mathbf{k}) = \pm |t' e^{ik\delta_1} + t e^{ik\delta_2} + t e^{ik\delta_3}|$$

G. Montambaux et al. (2009)

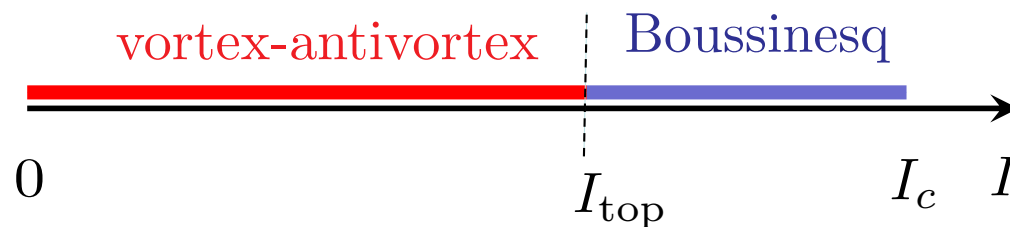
V. M. Pereira et al. (2009)



Topological transition between instantons



Second order transition appears most plausible



Summary

- First analytical solution for the GL instanton in 2D
- Obtained in the limit $I \rightarrow I_c$
 - Stream function
 - Elliptic Boussinesq equation
 - Hirota's form of the solution
- Second order topological transition expected at $I_{\text{top}} \approx 0.9 I_c$

MAS and A. V. Polkin, arXiv:2506.18130